Using Lyapunov exponents to predict the onset of chaos in nonlinear oscillators

Vladimir B. Ryabov
Complex Systems Department, Future University Hakodate, 116-2 Kameda Nakano-cho, 041-8655 Hakodate City, Hokkaido, Japan and Institute of Radio Astronomy, National Academy of Sciences of Ukraine, 4 Krasnoznamennaya Street, 310002 Kharkov, Ukraine
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An analytic technique for predicting the emergence of chaotic instability in nonlinear nonautonomous dissipative oscillators is proposed. The method is based on the Lyapunov-type stability analysis of an arbitrary phase trajectory and the standard procedure of calculating the Lyapunov characteristic exponents. The concept of temporally local Lyapunov exponents is then utilized for specifying the area in the phase space where any trajectory is asymptotically stable, and, therefore, the existence of chaotic attractors is impossible. The procedure of linear coordinate transform optimizing the linear part of the vector field is developed for the purpose of maximizing the stability area in the vicinity of a stable fixed point. By considering the inverse conditions of asymptotic stability, this approach allows formulating a necessary condition for chaotic motion in a broad class of nonlinear oscillatory systems, including many cases of practical interest. The examples of externally excited one- and two-well Duffing oscillators and a planar pendulum demonstrate efficiency of the proposed method, as well as a good agreement of the theoretical predictions with the results of numerical experiments. The comparison of the proposed method with Melnikov's criterion shows a potential advantage of using the former one at high values of dissipation parameter and/or multifrequency type of excitation in dynamical systems.

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I. INTRODUCTION

Lyapunov characteristic exponents (LCE) provide a quantitative measure of stretching and contracting deformations of an infinitesimally small phase space sphere in the vicinity of an arbitrary trajectory in a dynamical system. So defined, they also characterize the divergence (convergence) rates of two initially close trajectories residing on an attractor and serve as indicators of the stability of motion. Total number of Lyapunov exponents that a system possesses is equal to the dimension of the phase space, or, in other words, the number of independent variables necessary to fully characterize the motion. Being invariant under a smooth change of coordinates, LCE provide a useful quantitative measure of stability for various types of motion including complex quasiperiodic orbits and chaos and, together with other dynamic invariants such as fractal dimension and Kolmogorov-Sinai entropy [1], play an important role in the theory of nonlinear oscillations.

The fact that LCE can be used for distinguishing different types of attractors in dynamical systems makes them especially useful for the purpose of classifying the complicated oscillatory regimes or detecting transitions between motions of different types. In particular, positive sign of the largest LCE is commonly accepted as a hallmark of chaotic oscillations, which demonstrate strong sensitivity to initial conditions and exponential time diversion of nearby trajectories. On the contrary, stable periodic or quasiperiodic orbits are characterized by negative values of all the LCE (except the one in the tangent direction to the trajectory that is always zero). The knowledge of the full Lyapunov spectrum (ordered by their magnitude values of all LCE) is of crucial importance for understanding the basic phenomenology in many problems of mechanics [2], quantum physics [3], theory of turbulence [4], biological systems [5], or geophysics [6]. Systems with two or more positive LCE also attract considerable attention as examples of hyperchaotic behavior in view of their potential application for secure communication schemes based on chaotic synchronization [7].

It should be noted that, by their definition, LCE are asymptotic quantities defined in the limit of $t\rightarrow\infty$, as they characterize the average growth rate of a set of mutually orthogonal vectors in the tangent space. Their magnitudes generally depend on the starting point on the trajectory (initial conditions), but, if the motion on the attractor is ergodic, the averaged values are the same for almost all initial conditions in the basin of attraction, except, maybe, for a set of Lebesgue measure zero [1,8]. In 1968 it had been proven by Oseledec [9] that such long time averages do exist for a broad class of dynamical systems including most of the situations of practical interest.

In many cases it turned out to be constructive studying not only the mean values, but also the fluctuations of expansion rates in both time and phase space. The values of Lyapunov exponents calculated over a finite time interval depend on initial conditions, and corresponding distribution functions can be introduced as quantitative measures of complexity [8,10,11]. Such exponents are usually called local Lyapunov exponents (LLE) or local growth rates to account for their dependence on the position of the starting point in the phase space, and they have been proven useful in many works studying the statistical properties of strange attractors [12–14], attractor crises [10,11], intermittency [15], and time variation of the fractal dimension of strange attractors [16,17].

On the other hand, different exponents have been introduced to study both time and phase space variability of the stability exponents in the limit $t\rightarrow 0$. Mathematically, these exponents can be interpreted as limit case of LLE calculated within an infinitesimally small time interval. In order to distinguish these instantaneous growth rates from LLE and stress the importance of the explicit time dependence we call them *temporally local Lyapunov exponents* (TLLE). These quantities have been demonstrated to be effective in several ways, e.g., for characterizing the interaction between deterministically chaotic and noisy systems [18–20], quantifying local predictability in the phase space [21,22], or numerically calculating the values of traditional LCE [23,24].

In this paper we consider another application of TLLE, the method for predicting the emergence of general type of instability in dissipative dynamical systems that may cause the formation of a strange attractor with at least one positive LCE. Our approach is based on the possibility to derive explicit equations that govern the time evolution of TLLE, directly from the original set of differential equations describing the dynamical system. Then the equations for TLLE can be analyzed analytically, together with the set of governing equations, enabling one to obtain the estimate of the stability area in the phase space and/or the space of control parameters. Note that, although it is typically highly desirable to know the dependence of the Lyapunov spectrum on the control parameters, this problem defies analytical treatment. As a rule, it appears impossible to obtain the values of LCE from the functional form of the multidimensional mapping or system of nonlinear differential equations defining the evolution of the dynamical system of interest. So far, the numerical calculations remain the only straightforward way of analysis, when the information on the largest LCE or the full Lyapunov spectrum is necessary [1,25–27].

Contrary to previous studies focused mainly on the numerical analysis of LCE we develop a procedure for obtaining their estimates analytically from differential equations. The feasibility of the principal idea based on the analysis of dynamical equations for Lyapunov vectors in the tangent space (initially formulated in the context of systems of linear equations with periodic coefficients [28]) has been already demonstrated for several nonautonomous nonlinear oscillators with one-and-a-half degrees of freedom [29]. Here we generalize the results reported in Ref. [29] to a broad class of dynamical systems of arbitrary dimension and various types of nonlinearity. The ultimate condition for the emergence of instability is expressed in the form of stability criterion for a bounded region in the phase space, where all trajectories are asymptotically stable, and, therefore, converge to either fixed points or stable periodic attractors. The proposed way of analysis is fundamentally a Floquet's type approach [30] but generalized to include nonperiodic or multifrequency types of motion. Moreover, the final result formulated in terms of the amplitude of motion does not depend on the particular functional form of the external force. From this viewpoint, it could be especially useful in a situation of broadband external excitation [31,32] of oscillators, or noisy dynamical systems where the random time perturbations cannot be considered small.

The paper is organized as follows. In Sec. II we introduce the concept of TLLE for systems of differential equations and formulate the stability conditions for an arbitrary trajectory in the phase space. Then we discuss the effect of a linear coordinate transform on the stability properties of governing equations and describe a procedure for optimizing the stability criterion in terms of the amplitude and/or velocity of motion. Section III provides the analysis and explicit formulas for the stability conditions in an externally excited dynamical system of the second order. In Sec. IV the difference between linear and nonlinear systems is discussed from a stability viewpoint. The procedure is further developed for generalizing the stability analysis of linear system to the case of arbitrary degree of nonlinearity in the oscillators of the second order. Section V gives several examples of analysis for several classical nonlinear oscillators, such as Duffing system and mathematical pendulum. The comparison of the predictions for chaos onset made with the proposed technique to those following from other approaches, like Melnikov method or conventional stability analysis, has been also carried out. Section VI contains a summary and interpretation of results in terms of the necessary conditions of chaotic instability in dynamical systems.

II. MATHEMATICAL FRAMEWORK

A. Definition of temporally local Lyapunov exponents

TLLE are introduced in the following way [18–20,23]. Consider a dynamical system described by the set of n ordinary differential equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x},\ t),\quad \mathbf{x}\in\mathcal{R}^n. \tag{1}$$

Stability of an arbitrary solution of Eq. (1) $\mathbf{x}^*(t)$, is defined by the linearized system,

$$\frac{d\mathbf{y}}{dt}=\hat{\mathbf{J}}(\mathbf{x}^*(t))y, \tag{2}$$

where $\hat{\mathbf{J}}(\mathbf{x}^*(t))\equiv\partial\mathbf{F}(\mathbf{x}^*(t))/\partial\mathbf{x}$ is $n\times n$ time-dependent Jacobian matrix, $\mathbf{y}$ is an n vector in the tangent space corresponding to an infinitesimal perturbation of the trajectory $\mathbf{x}^*(t)$. The standard algorithm for calculating the spectrum of LCE [25,26] consists in solving Eqs. (2) simultaneously with Eq. (1) for a set of mutually orthonormal vectors $\{\mathbf{y}_k\}$ $(k=1,2,...,n)$ and estimating the average expansion rates for the lengths $\rho_k=\|\mathbf{y}_k\|$ of the vectors $\{\mathbf{y}_k\}$. The general solution of Eq. (2) is given by

$$\mathbf{y}(t)=\hat{\mathbf{M}}(t)\mathbf{y}(0),$$

where $\hat{\mathbf{M}}(t)$ is the fundamental matrix of solutions for Eq. (2). It follows from the results obtained by Oseledec [9] that for almost any choice of initial conditions there exists the following long time limit for the norms of a suitably chosen set of orthonormal vectors $\mathbf{y}_k(0)$:

$$\lambda_k=\lim_{t\to\infty}\frac{1}{t}\ln\|\hat{\mathbf{M}}(t)\mathbf{y}_k(0)\|. \tag{3}$$

In other words this means that asymptotically, in the limit of $t\to\infty$, the evolution of $\|\mathbf{y}_k\|$ is approximated by $\|\mathbf{y}_k(t)\|=\|\mathbf{y}_k(0)\|e^{\lambda_k t}$, where the exponents λ_k constitute the spectrum of LCE.

In order to obtain the dynamical equations for TLLE we use the approach similar to the one described in Refs. [21–24] and rewrite Eq. (2) in the polar coordinate frame for the amplitude $\rho=\|\mathbf{y}\|$ and directions φ_m $(m=1,2,...,n-1)$ of an arbitrary vector $\mathbf{y}$ in the tangent space.

$$\rho\frac{d\rho}{dt}=\sum_{l=1}^{n} y_l\frac{dy_l}{dt}, \tag{4a}$$

$$\frac{d\varphi_m}{dt}=\Phi(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}), \tag{4b}$$

where y_l are Cartesian components of the vector $\mathbf{y}$, $\rho^2=\Sigma_{l=1}^{n}y_l^2$; and the angles φ_m can be found from the following formulas defining the transition from Cartesian to spherical coordinates in $\mathcal{R}^n$:

$$y_1=\rho\cos\varphi_1,$$

$$y_2=\rho\sin\varphi_1\cos\varphi_2,$$

$$y_i=\rho\cos\varphi_i\prod_{l=1}^{i-1}\sin\varphi_l, \tag{5}$$

$$y_{n-1}=\rho\cos\varphi_{n-1}\prod_{l=1}^{n-2}\sin\varphi_l,$$

$$y_n=\rho\prod_{l=1}^{n-1}\sin\varphi_l.$$

It is easy to see that, e.g., in the case $n=3$ Eqs. (5) are reduced to the standard spherical coordinates in the three-dimensional space (ρ, φ, θ), where $\varphi_1=\theta$, $\varphi_2=\varphi$, and, hence, the formulas (5) are just a generalization of the standard spherical coordinates to a high-dimensional space.

Note that if we put $\rho(0)=1$ then by the definition (3) the spectrum of LCE is expressed as

$$\lambda_k=\lim_{T\to\infty}\frac{1}{T}\ln\rho_k(T), \tag{6}$$

where $\rho_k(T)$ correspond to the lengths of the initially orthonormal vectors $\{\mathbf{y}_k\}$ after the time interval T.

By dividing Eq. (4a) by ρ^2, we obtain the differential equation for the time evolution of $\ln\rho(t)$,

$$\frac{d}{dt}[\ln\rho(t)]=\frac{1}{\rho^2}\left[\sum_{l=1}^{n} y_l\frac{dy_l}{dt}\right]. \tag{7}$$

On the other hand, Eq. (2) rewritten in the scalar form for each of the Cartesian components of the vector $\mathbf{y}$ looks like

$$\frac{dy_l}{dt}=\sum_{m=l}^{n} j_{lm}(\mathbf{x}^*(t))y_m=\rho\sum_{m=l}^{n} j_{ln}(\mathbf{x}^*(t))\cos\varphi_m\prod_{i=1}^{m-1}\sin\varphi_i, \tag{8}$$

where j_{lm} are the components of the time-dependent $n\times n$ matrix $\mathbf{J}$, and Eqs. (5) have been also taken into account. Note that each component y_l in Eqs. (5), as well as its time derivative dy_l/dt, given by Eq. (8), is a linear function of ρ, therefore, the dynamics of $\ln[\rho(t)]$ do not depend on ρ, and the general form of the evolution equation for $\ln[\rho(t)]$ can be now written as

$$\frac{d}{dt}[\ln\rho(t)]=P(\varphi_1,\varphi_2,\ldots,\varphi_{n-1}), \tag{9}$$

where P is a function of angular coordinates only. Integration of Eq. (9) over the finite time interval T gives

$$\ln[\rho(T)]=\int_0^T P(\varphi_1(t),\varphi_2(t),\ldots,\varphi_{n-1}(t))dt, \tag{10}$$

where the time-dependent functions $\varphi_m(t)$ $(m=1,2,...,n-1)$ are defined by the solutions of Eqs. (4b) and (1). Eventually, by substituting Eq. (10) in Eq. (6) we obtain the following equations, defining the spectrum of LCE:

$$\lambda_k=\lim_{T\to\infty}\frac{1}{T}\int_0^T P_k(\varphi_1(t),\varphi_2(t),\ldots,\varphi_{n-1}(t))dt. \tag{11}$$

It follows from Eq. (11) that LCE are, in fact, long time averages of corresponding functions of angular coordinates of the vectors $\{\mathbf{y}_k\}$. If we calculate the integral in Eq. (11) over a finite time interval, we obtain the growth rates, depending on the starting point of integration in Eqs. (1), i.e., the spectrum of LLE. The instantaneous values of the functions $P_k(\varphi_1(t),\varphi_2(t),\ldots,\varphi_{n-1}(t))$ depend on both the time and phase space coordinates and constitute the spectrum of TLLE. So, the following expression can be used as the definition of TLLE [denoted hereafter as $\mu_k(t)$]:

$$\mu_k(t)=\frac{d[\ln\rho_k(t)]}{dt}=P_k(\varphi_1(t),\varphi_2(t),\ldots,\varphi_{n-1}(t)). \tag{12}$$

The angles φ_i $(i=1,2,...,n-1)$ in Eq. (12) have to be calculated from Eq. (4b) solved simultaneously with Eq. (1).

The equations for the angles in Eq. (4b), as well as the functions $P_k(\varphi_1(t),\varphi_2(t),\ldots,\varphi_{n-1}(t))$, do not depend on the amplitudes ρ_k and, hence, can be integrated independently from Eq. (4a). This assertion can be proved by differentiating Eqs. (5) with respect to time and substituting Eqs. (2) on their left-hand side. For an arbitrary component of the vector $\mathbf{y}$ we have from Eq. (5),

$$\frac{dy_i}{dt}=\frac{d\rho}{dt}\cos\varphi_i\prod_{l=1}^{i-1}\sin\varphi_l-\rho\frac{d\varphi_i}{dt}\sin\varphi_i\prod_{l=1}^{i-1}\sin\varphi_l+\rho\cos\varphi_i\frac{d}{dt}\left[\prod_{l=1}^{i-1}\sin\varphi_l\right], \tag{13}$$

where Π denotes the usual product. Now, starting from the last two components of the vector $\mathbf{y}$, y_{n-1} and y_n, we multiply their derivatives by $\sin(\varphi_{n-1})$ and $\cos(\varphi_{n-1})$, respectively, and subtract one from another. Finally, we have the implicit equation for the dynamics of the phase φ_{n-1},

$$\frac{dy_n}{dt}\cos\varphi_{n-1}-\frac{dy_{n-1}}{dt}\sin\varphi_{n-1}=\rho\frac{d\varphi_{n-1}}{dt}\prod_{l=1}^{n-2}\sin\varphi_l. \tag{14}$$

After substituting Eq. (8) for $l=n$, $n-1$ in Eq. (14) and dividing by ρ, we come to the equation for $d\varphi_{n-1}/dt$ of the form (4b), where the right-hand side depends only on angular coordinates. It can be proved by induction that all other equations for the angles φ_m in Eq. (4b) are also independent of ρ.

It should be noted that some of the just discussed general properties of Eqs. (4) defining the dynamics of arbitrary vectors in tangent space have been already utilized by other authors in different contexts. In particular, the independence of expansion rates on the length of the vector in the tangent space justifies the validity of the periodic renormalization of the lengths $\|\mathbf{y}_k\|$ used in the standard procedure of LCE computation for avoiding the overflow due to the exponential growth of $\|\mathbf{y}_1\|$ on a chaotic attractor [1,26]. On the other hand, the relations similar to Eq. (4) have been recently demonstrated to be useful for developing an efficient method of computing the LCE spectrum [24]. The approach proposed in Ref. [24] reduces the number of necessary-to-integrate equations in the system (2) by excluding the length of the vectors $\{\mathbf{y}_k\}$ from consideration and analyzing the dynamics of some suitably chosen angular variables only.

B. Stability of solutions of differential equations and TLLE

Any trajectory in the phase space of system (1) is stable, if all the corresponding LCE are nonpositive. Typically, the stability can be lost when at least one of the LCE calculated along the trajectory becomes positive due to the change in the control parameters. In terms of the tangent space, it means that the length of an arbitrary vector defining the perturbation starts growing exponentially with time. As we already noted, it is of particular importance to be able to predict the occurrence of such kind of transition in dynamical systems. In the general case of an arbitrary dynamical system it appears impossible to develop an analytical method that would allow solving the above problem. It is, therefore, highly desirable to obtain at least some estimates for the values of control parameters where the instability may occur, or, in other words, to derive a necessary condition for the chaotic motion to appear. In terms of stability theory, an inverse of the necessary condition of instability constitutes a sufficient condition for stability, therefore, one can always specify where the unstable motions can arise by finding an area of stability in the phase space or the space of control parameters.

As follows from (11), (12), the LCE (λ_i) are long time averages of the corresponding TLLE (μ_i). If we arrange the values of λ_i in descending order, then the instability means the positive value of the first (largest) LCE, i.e., $\lambda_1>0$. It is evident, that λ_1 can take a positive value only if μ_1 is greater than zero during some time intervals. On the contrary, if the inequality

$$\mu_1(t)<0 \tag{15}$$

holds all the time, the system is asymptotically stable, i.e., all the perturbations are exponentially shrinking with time and, hence, chaotic motions are precluded. From the inequality (15), together with Eqs. (1), (2), it appears possible to obtain the relation between the control parameters and phase space coordinates which guarantees that the system is "safe" in the sense that if the trajectory never leaves the region with negative values of μ_1, then no chaotic behavior appears. The goal is reached by analyzing the structure of the function $\mu_1(t)=P_1(\varphi_1(t),\varphi_2(t),\dots,\varphi_{n-1}(t))$, together with solutions of Eq. (1), which define the dynamics of angles φ_m through Eqs. (2) and (4). The general form of the function $P_1(\varphi_1(t),\varphi_2(t),\dots,\varphi_{n-1}(t))$ follows from Eqs. (2), (5), (7). If we substitute Eq. (2) in (7) and use Eq. (5), the following relations are obtained:

$$\begin{aligned}P_1(\varphi_1(t),\ \varphi_2(t),\dots,\varphi_{n-1}(t))&=\frac{1}{\rho^2}\sum_{l=1}^{n}\sum_{m=1}^{n}j_{lm}(\mathbf{x}^*(t))y_ly_m\\&=\sum_{l=1}^{n}\sum_{m=1}^{n}j_{lm}\cos\varphi_l\cos\varphi_m\prod_{i=1}^{l-1}\sin\varphi_i\prod_{k=1}^{m-1}\sin\varphi_k.\end{aligned} \tag{16}$$

The terms in the right-hand side of the Eq. (16) can be regrouped in a way allowing the conclusion on the sign of $P_1(\varphi_1(t),\varphi_2(t),\dots,\varphi_{n-1}(t))$ to be made,

$$P_1(\varphi(t,\mathbf{x}^*(t)))=G_1(\hat{\mathbf{J}}(\mathbf{x}^*(t)))+H_1(\varphi(t,\mathbf{x}^*(t)),\hat{\mathbf{J}}(\mathbf{x}^*(t))), \tag{17}$$

where the function G_1 does not depend on the angles φ_i and is defined by the dynamics of the diagonal elements of the perturbation matrix $\hat{\mathbf{J}}$ only,

$$G_1(\hat{\mathbf{J}})=\sum_{i=1}^{n-1}\frac{1}{2^n}j_{ii}(\mathbf{x}^*(t))+\frac{1}{2^{n-1}}j_{nn}(\mathbf{x}^*(t)). \tag{18}$$

The function H_1 can be expressed as the sum of products of matrix elements $j_{lm}(\mathbf{x}^*(t))$ and cosine or sine functions of various linear combinations of angles φ_l,

$$H_1(\varphi,\hat{\mathbf{J}})=\sum_{l,m}\frac{1}{c_{lm}}j_{lm}\sin\left(\sum_i k_i\varphi_i\right), \tag{19}$$

where c_{lm}, k_i are integer constants, j_{lm} and φ_i are time-dependent functions. The criterion of stability for any solution of Eq. (1) defined by inequality (15) can now be reformulated in terms of the functions H_1, G_1 as follows:

$$\min_t [G_1(\hat{\mathbf{J}}(\mathbf{x}^*(t)))] > \max_{t,\varphi} [H_1(\varphi(t,\mathbf{x}^*(t)),\hat{\mathbf{J}}(\mathbf{x}^*(t)))]. \tag{20}$$

The opposite inequality would mean the potential instability of the trajectory $\mathbf{x}^*(t)$ in the phase space of the system (1). Indeed, the stability of any trajectory is defined by the balance of positive and negative values of the largest TLLE. As the phase trajectory evolves in the phase space, during certain time intervals it can lie outside the border defined by Eq. (20), where the perturbations are exponentially amplified by the dynamics. If the positive values of TLLE prevail on average, the motion becomes unstable, including the possibility for chaotic attractors to appear. When the long-time average of the first TLLE is negative, then the largest LLE is also negative and, therefore, the whole trajectory is stable. Actually, the inequality (20) defines an area in the phase space where any trajectory is asymptotically stable. If the trajectory never leaves the area of stability (20) it cannot become unstable and, therefore, it is by no means chaotic.

It should be, however, noted that a straightforward calculation of the functions G_1 and H_1 from Eqs. (1), (2) does not always allow us to obtain the explicit equation for the border of the asymptotic stability area in the phase space. As we demonstrate below with several examples of nonlinear oscillators this happens due to the presence of both the expanding and contracting directions around a typical trajectory that is a consequence of the affine character of the phase flow in the vicinity of a generic stable fixed point (see, Fig. 1). In terms of the functions G_1 and H_1 this means that $\min_t[G_1(\hat{\mathbf{J}}(\mathbf{x}^*(t)))]$ may be strictly less than $\max_{t,\varphi}[H_1(\varphi(t,\mathbf{x}^*(t)),\hat{\mathbf{J}}(\mathbf{x}^*(t)))]$ for any trajectory $\mathbf{x}^*(t)$, and, hence, the inequality (20) may never be satisfied.

Fortunately, the particular form of the functions G_1 and H_1 depends on the choice of coordinates (see also Ref. [33]), and in many cases it turns out possible to obtain the borders of the asymptotic stability area by introducing a suitable coordinate transformation. As will be shown below, the universal solution to the problem of finding such a transform is provided by a linear change of coordinates diagonalizing the linear part of the flow $\mathbf{F}(\mathbf{x},t)$ in the vicinity of an arbitrary point in the phase space. This kind of transformation is known to be a standard tool in the analysis of differential equations [see, e.g., [34]] and is usually used as a first step allowing us to simplify the linear part of the problem "as much as possible."

C. Linear coordinate transform

A standard way of analyzing a dynamical system can be roughly described as follows. The system (1) can be represented by the form

$$\frac{d\mathbf{x}}{dt} = \hat{\mathbf{A}}\mathbf{x} + \varepsilon \mathbf{N}(\mathbf{x}) + \mathbf{f}(t), \quad \mathbf{x} \in \Re^n, \tag{21}$$

where $\hat{\mathbf{A}}$ is a constant $n \times n$ matrix, $\mathbf{N}(\mathbf{x})$ is a nonlinear vector-function vanishing at the origin $\mathbf{N}(\mathbf{0}) = 0$, together with all of its partial derivatives

$$\left. \frac{\partial \mathbf{N}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=0} = 0,$$

ε is a dimensionless parameter, not necessarily small, $\mathbf{f}(t)$ is a vector of external forces. Such problem posing implies the presence of a fixed point at the origin when $\mathbf{f}(t) \equiv 0$. In the case of a different position of the fixed point, it can be always shifted to zero by the change of coordinates $\mathbf{x} \to \mathbf{x} - \mathbf{x}_0$, which moves the origin to the location $\mathbf{x}_0$ of the fixed point. Moreover, we also assume that the fixed point is stable, i.e., all the eigenvalues of the matrix $\hat{\mathbf{A}}$ have negative real parts.

A linear coordinates transform

$$\mathbf{z} = \hat{\mathbf{B}}\mathbf{x}, \tag{22}$$

where $\hat{\mathbf{B}}$ is a constant real $n \times n$ matrix, recasts the system (21) to the form

$$\frac{d\mathbf{z}}{dt} = \hat{\mathbf{B}}\hat{\mathbf{A}}\hat{\mathbf{B}}^{-1}\mathbf{z} + \varepsilon \hat{\mathbf{B}}\mathbf{N}(\hat{\mathbf{B}}^{-1}\mathbf{z}) + \mathbf{g}(t), \quad \mathbf{z} \in \Re^n, \tag{23}$$

where, the vector-function $\mathbf{g}(t)$ defines the external forces in the new coordinates. After the transformation (22), the linearized equations of motion read

$$\frac{d\mathbf{w}}{dt} = \hat{\mathbf{L}}(\mathbf{z}^*(t))\mathbf{w}, \tag{24}$$

where the elements of the perturbation matrix $\hat{\mathbf{L}}$ now consist of two parts

$$\hat{\mathbf{L}}(\mathbf{z}^*(t)) = \hat{\mathbf{D}} + \varepsilon \hat{\mathbf{B}} \left. \frac{\partial}{\partial \mathbf{z}} [\mathbf{N}(\hat{\mathbf{B}}^{-1}\mathbf{z})] \right|_{\mathbf{z}=\mathbf{z}^*(t)}, \tag{25}$$

where the notation $\hat{\mathbf{D}} \equiv \hat{\mathbf{B}}\mathbf{A}\hat{\mathbf{B}}^{-1}$ is used. The elements of the matrix $\hat{\mathbf{B}}$ can be always chosen in a way making the matrix $\hat{\mathbf{D}}$ diagonal or block diagonal [35]. This coordinate transform makes the fixed point at the origin of the tangent space not affine, that results in the absence of expanding directions for a vector centered at the origin. As a consequence, after the transformation (22) diagonalizing the linear part of the problem, there appears a certain area in the phase space, where any infinitesimal sphere experiences only contraction in any direction. Then, the inequality (20) in the new coordinates allows obtaining an explicit equation for its border.

It should be, however, noted that after such a transformation of coordinates, the condition (20) might provide not the best possible estimate for the size of the asymptotic stability area. Under any circumstances, it gives some approximation for, e.g., the value of the maximal stable amplitude of motion or maximal velocity. However, since the analysis used for diagonalizing the matrix $\hat{\mathbf{A}}$ utilizes the information on the properties of the linear part of the problem only, the obtained estimate of the size of stability area is, as a rule, strongly underestimating the values of coordinates and/or control parameters where chaotic instability is likely to occur. As we demonstrate below, a substantial improvement can be achieved by applying a different coordinate transform of the type (22). The modification consists in taking into account the particular form of the nonlinearity function $\mathbf{N}(\mathbf{x})$, when calculating the elements for the transformation matrix $\hat{\mathbf{B}}$. Then, in the new coordinates, the linear part of the system is nondiagonal, but, nevertheless, an infinitesimal sphere in the tangent space contracts in all directions. The advantage of the proposed method consists in maximizing the size of the asymptotic stability area, or, in a different perspective, obtaining a better estimate for the chaotic instability threshold in terms of the amplitude and/or velocity of motion.

In subsequent sections we demonstrate the feasibility of the proposed method with several examples of nonautonomous nonlinear oscillators of the second order. Although the technique can be used for the analysis of almost any (including high-dimensional) passive dynamical system, we would like to restrict our consideration in the present paper by the case of second-order nonautonomous systems (three-dimensional phase space). This choice enables us to compare our results with numerous estimates of chaotic instability area for these particular models existing in the literature.

III. ASYMPTOTIC STABILITY CRITERION FOR A NONAUTONOMOUS SYSTEM OF THE SECOND ORDER

Evolution equations (1) for an arbitrary dynamical system of the second order with an external force are given by

$$dx_1/dt=f_1(x_1;x_2)+F_1(t), \quad dx_2/dt=f_2(x_1;x_2)+F_2(t), \tag{26}$$

where the nonlinear functions $f_{1,2}$ are defined by the properties of the system, $F_{1,2}(t)$ characterize the applied forces, acting independently of the system dynamics. Following the procedure developed in the previous section we rewrite the linearized equations (2) for an arbitrary vector $\mathbf{y}$ in the tangent space in the polar coordinate frame. Finally, we obtain the equations describing the dynamics of the length ρ and phase φ of the vector $\mathbf{y}$

$$\frac{d\rho}{dt}=\frac{\rho}{2}[j_{11}+j_{22}+(j_{11}-j_{22})\cos(2\varphi)+(j_{12}+j_{21})\sin(2\varphi)],$$

$$\frac{d\Phi}{dt}=\frac{1}{2}[j_{21}-j_{12}+(j_{12}+j_{21})\cos(2\varphi)+(j_{22}-j_{11})\sin(2\varphi)], \tag{27}$$

where $j_{ij}(\mathbf{x}^*(t))\equiv\partial f_i/\partial x_j$ are the components of the perturbation matrix $\hat{\mathbf{J}}$ depending on time through the solutions $\mathbf{x}^*(t)$ of Eqs. (26). The expression for the largest TLLE follows directly from the first of the Eqs. (27) and definition (12)

$$2\mu_1(t)=j_{11}+j_{22}+(j_{11}-j_{22})\cos(2\varphi)+(j_{12}+j_{21})\sin(2\varphi), \tag{28}$$

where the time dependence of μ_1 is defined by the concurrent solution of Eqs. (26) and (27). The condition for the asymptotic stability of the solution $\mathbf{x}^*(t)$ of the system (21) can be now obtained from Eqs. (20), (28) as

$$\min_{\mathbf{x}^*(t)}[(j_{11}+j_{22})^2]>\max_{\mathbf{x}^*(t)}[(j_{11}-j_{22})^2+(j_{12}+j_{21})^2]. \tag{29}$$

In the following, when studying the inequalities of the type (29), we shall omit the min and max functions calculated over the range of variation of the phase space coordinates. However, it should be kept in mind that an arbitrary trajectory is asymptotically stable only if the inequality of type (29) is satisfied within a certain range of the phase space coordinates (from minimal to maximal values) calculated along the corresponding solutions of Eqs. (26).

Since the matrix elements j_{ik} depend on coordinates (x_1,x_2), the inequality (29) defines the area in the phase space where the perturbations around a fiducial trajectory contract with time exponentially, thus ensuring the asymptotic stability of the solution. Note also that the left-hand side of Eq. (29) is the divergence of the phase space flow along the trajectory $\mathbf{x}^*(t)$, characterizing the overall dissipation properties of the system. From this perspective, the inequality (29) establishes a well-known fact that dissipation stabilizes the motion and imposes a threshold for the appearance of instabilities.

As we already noted, in a generic situation, it appears necessary to perform a change of coordinates, which transforms the inequality (29) to the form suitable for the subsequent analysis. An arbitrary linear transformation of type (22) does not change the general form of Eq. (28), but j_{ik} are now to be replaced with the corresponding elements of the matrix $\hat{\mathbf{L}}\equiv\hat{\mathbf{B}}\hat{\mathbf{J}}\hat{\mathbf{B}}^{-1}$ defined by the Eq. (25). The condition of asymptotic stability (29) can now be rewritten as $(l_{11}-l_{22})^2+(l_{12}+l_{21})^2<(l_{11}+l_{22})^2$ or, equivalently,

$$\sum_{i,k} l_{ik}^2<[\mathrm{Tr}(\hat{\mathbf{L}})]^2+2\det(\hat{\mathbf{L}}), \tag{30}$$

where l_{ik} are the elements of the matrix $\hat{\mathbf{L}}$, depending on phase space coordinates, control parameters, and elements of the transformation matrix $\hat{\mathbf{B}}$. Note, that the right-hand side of the inequality (30), being expressed via the trace and determinant of the matrix $\hat{\mathbf{L}}$, is invariant under any linear transformation and, hence, independent from the values of the matrix elements of $\hat{\mathbf{B}}$.

The sum at the left of Eq. (30) is the Euclidean norm of the matrix $\hat{\mathbf{L}}$, generally known to be dependent on the choice of coordinates [36]. It reaches a minimum, when $\hat{\mathbf{B}}$ transforms the matrix $\hat{\mathbf{L}}$ to the canonical (diagonal) form, and is unlimited from above. It is a well-established fact in the matrix theory that the Euclidean norm does not depend on rotations of the coordinate frame, or, in other words it is a unitary invariant matrix measure. Therefore, not all the elements of the matrix $\hat{\mathbf{B}}$ are independent parameters of the problem, and the inequality (30) can be recast to the following form containing two independent parameters only:

$$[u(j_{11}-j_{22})+j_{21}-(u^2+v^2)j_{12}]^2<4v^2(j_{11}j_{22}-j_{21}j_{12}), \tag{31}$$

where the new parameters u, v are introduced as

$$u=\frac{b_{11}b_{12}+b_{21}b_{22}}{b_{12}^2+b_{22}^2},\quad v=\frac{b_{11}b_{22}-b_{21}b_{12}}{b_{12}^2+b_{22}^2} \tag{32}$$

and j_{ik}, b_{ik} are the elements of the matrixes $\hat{\mathbf{J}}$ and $\hat{\mathbf{B}}$, respectively.

Since the values of the parameters u, v are arbitrary, they can be chosen in a way maximizing the size of the stability area in the phase space. The matrix elements j_{ik} are different nonlinear functions of the phase space coordinates x_1, x_2, and the range of their variation is defined by the particular functional form of the nonlinearity functions f_1, f_2 in Eqs. (26), the type of external perturbations F_1, F_2 and, eventually, by the size of the attractor in the phase space. So, if it is necessary to obtain the asymptotic stability conditions in terms of the control parameters, one has to estimate the range of variation of the coordinates x_1, x_2. It should be, however, noted that this problem cannot be solved for any type of external force and requires additional methods of analysis to be used. Below, we restrict our consideration by several examples and demonstrate the efficiency of the proposed approach for predicting the onset of chaos in these systems.

IV. NONAUTONOMOUS PASSIVE NONLINEAR OSCILLATOR

A. Asymptotic stability condition

As an example of a particular system with one-and-a-half degrees of freedom governed by the equations of the type (21), we take the following nonlinear oscillator:

$$\frac{d^2x}{dt^2}+\delta\frac{dx}{dt}+\omega_0^2x+\varepsilon N(x)=f(t) \tag{33}$$

that has been used as a basic model in many problems of mechanics, electronics, optics, electromagnetic field theory [31,34,37,38], etc. Here, x is a generalized coordinate, $\delta>0$ is a linear dissipation parameter, $N(x)$ is the function defining the shape of the potential well $[N(0)=0;\ N'(0)=0$, where prime means differentiation by $x]$, ε is the dimensionless parameter, and $f(t)$ is the external force. Note that Eq. (33) includes such classical systems as Duffing oscillator and mathematical pendulum as special cases.

By introducing the variables $x_1=x$; $x_2=dx/dt$ the system (33) is transformed to the standard form

$$\frac{dx_1}{dt}=x_2,$$

$$\frac{dx_2}{dt}=-\delta x_2-\omega_0^2x_1-\varepsilon N(x_1)+f(t) \tag{34}$$

and the variational equations (2) in the vicinity of an arbitrary trajectory $x^*(t)$ for this system look like

$$\frac{dy_1}{dt}=y_2,$$

$$\frac{dy_2}{dt}=-\omega_0^2y_1-\delta y_2-\varepsilon V(x^*(t))y_1, \tag{35}$$

where $y_{1,2}$ are the components of the perturbation vector $\mathbf{y}$,

$$V(x^*(t))=\left.\frac{dN}{dx_1}\right|_{x_1=x^*(t)}.$$

Then, the explicit expression for the largest TLLE follows directly from its definition (12) and the equation for the norm of $\|\mathbf{y}\|=\rho$ in the polar coordinates defined by $y_1=\rho\cos(\varphi)$; $y_2=\rho\sin(\varphi)$

$$\begin{aligned}\mu_1(t)&\equiv\frac{1}{\rho}\frac{d\rho}{dt}\\&=\frac{1}{2}[-\delta+\delta\cos 2\varphi(t)\\&\quad+(1-\omega_0^2-\varepsilon V(x^*(t)))\sin 2\varphi(t)],\end{aligned} \tag{36}$$

where $x^*(t)$ is an arbitrary solution of Eq. (33) and $\varphi(t)$ stands for the direction of the vector $\mathbf{y}$ defined by the Eqs. (35).

The functions $G_1(\hat{\mathbf{J}})$ and $H_1(\varphi,\hat{\mathbf{J}})$ defining the area of asymptotic stability in accordance with Eqs. (20), (29) can be easily found from Eq. (36) as

$$G_1(\hat{\mathbf{J}})=-\delta,$$

$$H_1(\varphi,\hat{\mathbf{J}})=\delta\cos(2\varphi)+(1-\omega_0^2-\varepsilon V(x^*))\sin(2\varphi). \tag{37}$$

It is clear from Eq. (37) that $\max_\varphi[H_1(\varphi,\hat{\mathbf{J}})]=\sqrt{\delta^2+(1-\omega_0^2-\varepsilon V(x^*))^2}$, which is always greater than $\min_t G_1(\hat{\mathbf{J}})=-\delta$, and, therefore, no conclusion about the stability area in the phase space can now be obtained. As we already mentioned, at this stage it is necessary to introduce a certain transformation of coordinates that would make the vicinity of origin not affine and enable one to obtain constructive results from these equations.

B. Normal form of a linear oscillator

In order to introduce the coordinate transform making feasible the stability analysis, we propose to consider first the most elementary case of a linear oscillator $(\varepsilon=0)$. Although such an analysis is rather trivial, it allows us to develop an intuition, necessary for a general case of oscillators with arbitrary degree of nonlinearity. So, just for the matter of clarity, we dwell upon this simple case in some more detail.

For the linear system, the variational equations (35) do not depend on the solutions of Eq. (34), and it is well known that for $\delta>0$, irrespective of the particular form of the time dependence of the external force, any trajectory is asymptotically stable, and no instability can appear. At the same time,

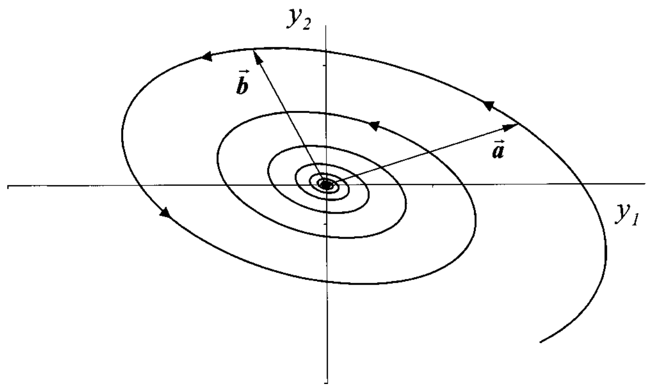

FIG. 1. The phase portrait of a linearized system in the vicinity of an arbitrary point of focus type.

the expression for the largest TLLE for this oscillator shows the presence of potential instability in exactly the same manner as it was in the case of the nonlinear oscillator (33)

$$\begin{aligned}\mu_1(t)&=\tfrac{1}{2}[-\delta+\delta\cos(2\varphi)+(1-\omega_0^2)\sin(2\varphi)]\\&=\tfrac{1}{2}[-\delta+A\cos(2\varphi-\psi)],\end{aligned}\tag{38}$$

where $A=\sqrt{\delta^2+(1-\omega_0^2)^2}$, $\psi=\tan^{-1}([1-\omega_0^2]/\delta)$. It follows from Eq. (38), that, since $A>\delta$, the phase of the vector **y** may hit the angular sector defined by $|2\varphi-\psi|<\cos^{-1}(\delta/A)$ where the largest TLLE is positive and, hence, the length of **y** may grow exponentially with time, thus presenting quite unexpected behavior for a linear system. This "unstable" character of solutions can be easily understood by considering the geometry of trajectories in the vicinity of the origin for the linearized system (35) shown in Fig. 1. If $0<\delta<2\omega_0$, there is a stable focus at the origin, whereas for $\delta>2\omega_0$ the focus becomes a stable node. Although, on average, the length of an arbitrary vector at the origin contracts with time, there are both stretching and contracting phases in its time evolution. For example, the vector $\vec{b}$ in Fig. 1 is stretched by the dynamics, while the vector $\vec{a}$ is getting shorter with time. This effect is caused by affine character of the phase flow, and it is a generic property of the phase flow in the vicinity of any stable fixed point. It is, however, well known from the theory of differential equations [35] that any linear phase flow can be made not affine by means of a linear coordinate transformation (22), which has no effect on the stability properties of trajectories [37] but recasts Eqs. (34), (35) to the form with the largest TLLE being strictly negative in a certain area of the phase space. After such a transformation, the equations (34) with $\varepsilon=0$ take the so-called normal form

$$\left.\begin{aligned}\frac{dz_1}{dt}&=\lambda_1 z_1+g_1(t)\\\frac{dz_2}{dt}&=\lambda_2 z_2+g_2(t)\end{aligned}\right.\qquad\text{for}\quad\delta^2>4\omega_0^2\tag{39a}$$

or

$$\left.\begin{aligned}\frac{dz_1}{dt}&=-\alpha z_1+\omega z_2+h_1(t)\\\frac{dz_2}{dt}&=-\omega z_1-\alpha z_2+h_2(t)\end{aligned}\right.\qquad\text{for}\quad\delta^2<4\omega_0^2,\tag{39b}$$

where Eqs. 39(a) and 39(b) correspond to the cases of node and focus types of fixed point at the origin, respectively. The roots of the characteristic polynomial of the system (34) with $\varepsilon=0$ defining the stability of the fixed point at the origin are real and negative for the node-type fixed point $\lambda_{1,2}=-(\delta/2)\pm\sqrt{\delta^2/4-\omega_0^2}$ and complex conjugates with negative real part for the focus-type fixed point: $\lambda_{1,2}=-(\delta/2)\pm i\sqrt{\omega_n^2-\delta^2/4}\equiv-\alpha\pm i\omega$. Furthermore, Eq. (38) for the largest TLLE now reads as

$$2\mu_1(t)=-\delta+\sqrt{\delta^2-4\omega_0^2}\cos(2\varphi),\tag{40a}$$

$$2\mu_1(t)=-\delta\tag{40b}$$

for the node and focus cases, respectively. It immediately follows from Eq. (40) that μ_1 is always strictly negative and in the case of a focus-type fixed point even does not depend on time. Therefore, we come to a conclusion that if $\varepsilon=0$, then, as it should be expected for any passive linear oscillator, all LCE are negative and any solution of Eqs. (34) is asymptotically stable. In the next section we derive similar conditions for the general case of an oscillator containing nonlinear terms. The basic scheme is essentially the same, except we demonstrate that, depending on the particular type of nonlinearity, somewhat different choice of the coordinate transform (22), may provide a better result maximizing the size of the stability area.

C. Stability of a nonlinear oscillator

Let us now consider what happens if nonlinearity is present, i.e., in the case of Eq. (33) at $\varepsilon\neq0$. Note that we do not impose any restrictions on the value of the parameter ε, therefore, an oscillator under study has an arbitrary degree of nonlinearity. We also assume that it possesses a stable fixed point at the origin in the absence of an external force. The latter assumption imposes a certain restriction on the class of the systems amenable to this type of analysis, although in many cases it can be avoided by a trivial change of coordinates shifting the position of the origin to that of one of the stable fixed points [34].

As we already noted, the straightforward stability analysis in terms of TLLE results in the equation of type (36), which gives no information on the size of asymptotic stability area. In the previous section we managed to derive a constructive result for a similar situation by applying the linear coordinate transform, reducing the system to a diagonal form. Proceeding in exactly the same way, by diagonalizing the linear part of the problem (34), one can recast Eq. (36) to the following form:

$$\mu_1(t)=\frac{1}{2}\left[-\delta+\left(\sqrt{\delta^2-4\omega_0^2}+\frac{2\varepsilon}{\sqrt{\delta^2-4\omega_0^2}}V(x^*(t))\right)\cos(2\varphi)\right] \quad \text{if} \quad \delta^2>4\omega_0^2, \tag{41a}$$

$$\mu_1(t)=\frac{1}{2}\left[-\delta-\frac{2\varepsilon}{\sqrt{4\omega_0^2-\delta^2}}V(x^*(t))\cos(2\varphi)\right] \quad \text{if} \quad \delta^2<4\omega_0^2 \tag{41b}$$

which, under the assumption that φ can take any value in the interval $[0;2\pi]$, leads to the explicit formulas of type (20) for the border of the asymptotic stability area

$$V_1<\varepsilon V(x)<V_2, \tag{42a}$$

where

$$V_{12}=\tfrac{1}{2}\sqrt{\delta^2-4\omega_0^2}\left(\sqrt{\delta^2-4\omega_0^2}\mp\delta\right) \quad \text{if} \quad \delta^2>4\omega_0^2, \tag{42b}$$

$$V_{1,2}=\mp\tfrac{1}{2}\delta\sqrt{4\omega_0^2-\delta^2} \quad \text{if} \quad \delta^2<4\omega_0^2. \tag{42c}$$

Equations (42) thus define the limits of variation for the function $V(x)$ and, hence, for the coordinate x of the nonlinear oscillator (33), ensuring the asymptotic stability of motion.

The conditions (42) are sufficient for the stability of all the trajectories in the corresponding area, but they are not necessary, in the sense that the areas of stable motion in the phase space may be larger, compared to those defined by inequalities (42). Indeed, if instead of making the coordinate transformation diagonalizing the linear part of the problem we consider the case of an arbitrary linear change of coordinates (22), Eqs. (42b), 42(c) for the borders of asymptotic stability take the form

$$V_{1,2}=[\sqrt{u(\delta-u)}\mp v]^2-\omega_0^2, \tag{43}$$

where u, v are the free parameters defined by the Eq. (32). Note that Eqs. (42b), 42(c) are just a special case of the Eq. (43). If we put $u=\delta/2$; $v=\sqrt{\pm(\delta^2-4\omega_0^2)}$, then the transformation (22) makes the linear part of the problem diagonal (for the cases of node or focus, depending on the sign in the expression for v), and Eq. (43) turns into Eqs. (42b), 42(c).

The size of the asymptotic stability area in the general case of arbitrary linear coordinate transformation is defined by the residual

$$V_2-V_1=4v\sqrt{u(\delta-u)}, \tag{44}$$

so the stability range of the function $V(x)$ is determined by the value of the dissipation parameter δ and parameters u, v, defined in their turn by the elements of the matrix $\hat{\mathbf{B}}$ of the coordinate transform (22). The values of u, v, however, are not absolutely arbitrary. The limitation consists in the necessity of including the origin in the interval of $V(x)$ variation, since, by our initial definition of the system (33), the function $V(x)$ had been chosen vanishing at the origin in order to split the original vector flow to linear and nonlinear subsystems. This condition determines the value for one of the parameters u, v. The remaining freedom in the choice of u, v can be further used in a constructive manner for maximizing the area of asymptotic stability in the phase space. The solution of the latter problem depends on the particular form of the nonlinear function $N(x)$, therefore, it should be performed on the case-by-case basis. Below, we demonstrate the efficiency of the proposed method for several nonlinear oscillators of the type (33).

V. EXAMPLES

A. Duffing oscillator with hardening type of nonlinearity function

To make the ideas developed in the previous section precise, let us specify the nonlinear function $V(x)$ and consider first the case of Duffing oscillator, i.e., Eq. (33) with

$$N(x)=x^3; \quad V(x)\equiv\frac{dN}{dx}=3x^2. \tag{45}$$

Many authors have studied this oscillator in different contexts (see, e.g., the books [39,40]). It can be considered as a classical example of a time-continuous nonlinear system capable of producing many types of complex behavior, including chaotic motion. As is well known [37], the main source of complex behavior in a generic dynamical system is saddle-type fixed points or periodic saddle orbits, which possess invariant manifolds capable of intersecting under the action of perturbation and forming homoclinic tangles. Since in the absence of external force there is only one fixed point of focus-type in the phase space of the Duffing oscillator, the chaotic behavior and other complex motions can appear here only after new saddle-type orbits are created by perturbation, for example, by a periodic external force. The absence of saddle points in the phase space of the unperturbed oscillator is, perhaps, the main reason for the lack of analytical methods allowing one to predict the onset of chaos in this system.

As an approximate criterion for the appearance of chaotic attractors, one can use the condition for saddle orbits to be created by external force or the stability loss of the existing periodic attractors. If the functional form of the external force is simple, like a harmonic or quasiperiodic function of time, this problem can be solved analytically, at least for small values of nonlinearity and dissipation parameters, by, e.g., methods of harmonic balance [30], averaging [41], and Floquet stability analysis. It, however, appears quite problematic (if possible at all) to predict the appearance of saddle-type orbits, when the number of harmonic components in the external force becomes larger than three. The problem also defies analytical treatment, when the dissipation parameter δ or the parameter of nonlinearity ε cannot be considered small. So, under such circumstances, a numerical experiment becomes the only tool of analysis for the study of various bifurcations and predicting the onset of chaotic oscillations in particular.

The method proposed in this paper is free from the drawbacks just mentioned, for it is developed for the *arbitrary* level of nonlinearity and dissipation, as well as *any* functional form of the external forcing. Indeed, by substituting Eq. (45) in (43), one can obtain the equations for the border of asymptotic stability in terms of the amplitude of motion as

$$x_{\min}^2=\frac{1}{3\varepsilon}[(\sqrt{u(\delta-u)}-v)^2-\omega_0^2], \tag{46a}$$

$$x_{\max}^2=\frac{1}{3\varepsilon}[(\sqrt{u(\delta-u)}+v)^2-\omega_0^2]. \tag{46b}$$

Equations (46) define minimal and maximal values for the amplitude of oscillations with guaranteed stability. They possess two free parameters, u and v, which are defined by the elements of the transformation matrix $\hat{\mathbf{B}}$. In order to specify their values, some additional information on the type of external force and character of motion is necessary. If we assume that the external force does not contain constant terms shifting the equilibrium position from the origin, and the motion occurs in an approximately symmetric area around the unperturbed fixed point, then we can put $x_{\min}=0$ and exclude the value of one of the parameters u, v from Eq. (46a) as

$$v=\sqrt{u(\delta-u)}+\omega_0. \tag{47}$$

After the substitution (47), the Eq. (46b) becomes

$$x_{\max}^2=\frac{4}{3\varepsilon}[\sqrt{u(\delta-u)}(\sqrt{u(\delta-u)}+\omega_0)]. \tag{48}$$

The demand of maximizing the size of the stability area leads to the choice of $u=\delta/2$, for Eq. (48) has a maximum at this point. Finally, we have the equation for the border of the stability area,

$$|x|<\sqrt{\delta(\delta+2\omega_0)/3\varepsilon}. \tag{49}$$

The general observation derived from the inequality (49) is that the higher the dissipation level the larger is the area of asymptotic stability around the origin. An example of the curve defined by the Eq. (49) is shown in Fig. 2 (heavy line) where we plot the dependence of the maximal stable amplitude versus dissipation parameter δ, when other parameters are fixed. In the same figure, we also plot the line of the maximal stable amplitude obtained with the coordinate transform $\hat{\mathbf{B}}$ diagonalizing the linear part of the problem, i.e., Eq. (42) (light line). One can see the advantage of using the optimized coordinate transform, resulting in a significant increase in the size of the asymptotic stability area. Another interesting fact is that higher values of natural frequency also result in a larger stability area, that means better stability to external perturbations of high-frequency oscillators compared to low-frequency ones.

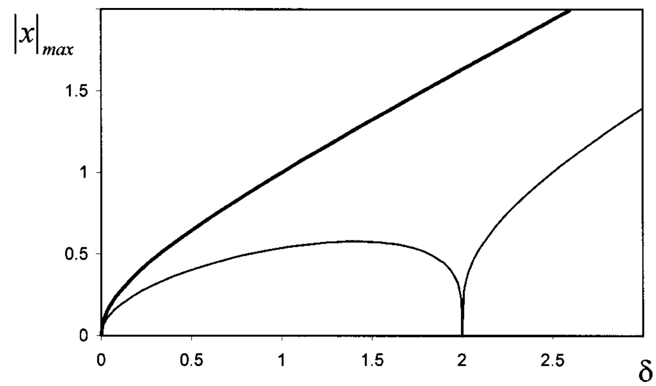

FIG. 2. Size of the asymptotic stability area vs dissipation parameter δ for the hard-mode Duffing oscillator at $\omega_0=1$, $\varepsilon=1$. The application of TLLE together with optimized linear coordinate transform results in a larger area of asymptotic stability (heavy line) compared to the case of the system with diagonalized matrix of the linear part (light line).

B. Duffing oscillator with soft nonlinearity

The equation of motion for this oscillator is similar to the case of hard-mode Duffing equation considered in the previous section, except the sign of the nonlinear term. Therefore, we have the equation of the type (33) with

$$N(x)=-x^3, \quad V(x)=-3x^2.$$

The change in the shape of nonlinearity function results in qualitatively different types of solutions typical of this oscillator. The motion now occurs not in the unbounded potential well as it was in the hard-mode oscillator, but in the potential shown in Fig. 3. In this case the amplitude of oscillations is limited from above by the homoclinic trajectory corresponding to the maximum of the potential curve, and the motion becomes unbounded when the trajectory crosses the line separating the area inside the potential well from that of the unbounded motion. The presence of saddle points and the separatrix in the unperturbed potential shown in Fig. 3 makes this system subject to various instabilities at substantially lower levels of perturbation compared to the Duffing oscillator with hard nonlinearity. It is well known that in the softly nonlinear oscillator chaotic motions appear at much

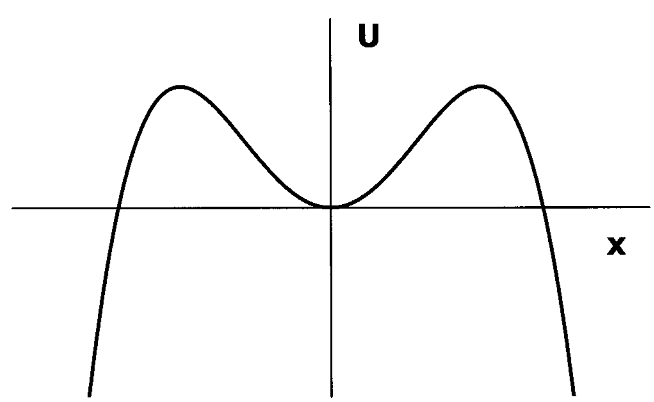

FIG. 3. Potential function for Duffing equation with soft-type nonlinearity.

lower values of the amplitude of external force, given that under all other parameters are kept constant. Usually, this fact is attributed to the presence of a homoclinic structure that arises in the phase space in the vicinity of the separatrix, when it becomes broken by the perturbation.

It is interesting that the approach we consider in this paper also allows us to detect the effect of lowering the instability threshold with respect to the amplitude of external force in the soft-mode oscillator. It should be, however, noted that the analysis of TLLE provides the threshold of stability in terms of the amplitude of oscillations, rather than the external forcing. This requires additional methods to be used, for establishing the relation between the amplitude of perturbation and response of the system.

To demonstrate how the method works for this system, we start from the observation that for the soft-type nonlinearity, the function $V(x)=-3x^2$ decreases with the growth in the oscillation amplitude. This results in the following restriction for the parameters u, v defined by the necessity of including the origin in the range of variation in the coordinate x,

$$v=\omega_0-\sqrt{u(\delta-u)}. \tag{50}$$

Then, the following equation for the maximal amplitude of motion can be obtained:

$$x_{\max}^2=\frac{4}{3\varepsilon}[\sqrt{u(\delta-u)}(\omega_0-\sqrt{u(\delta-u)})]. \tag{51}$$

Simple analysis of the Eqs. (50), (51) reveals that, depending on the value of δ, there may be two ways to choose the parameter u and, hence, the value of the maximal amplitude:

(1) If $\delta<\omega_0$, then maximal size of the stability area is attained at $u=\delta/2$, and

$$x_{\max}^2=\frac{\delta}{3\varepsilon}(2\omega_0-\delta). \tag{52}$$

(2) If $\delta>\omega_0$, then the parameter u has to be chosen as a root of the equation $u(\delta-u)=\omega_0^2/4$, and the maximal stable amplitude does not depend on δ,

$$x_{\max}^2=\frac{\omega_0^2}{3\varepsilon}. \tag{53}$$

An example calculation of the value of maximal stable amplitude for the soft-type Duffing equation is given in Fig. 4. The effect of optimizing the coordinate transform is not so pronounced here as it was in the case of hard-type nonlinearity, although the stability area is somewhat larger in optimal coordinates for this oscillator too. The comparison of Figs. 2 and 4 indicates that maximal stable amplitude is larger for the hard-mode oscillator, that is consistent with previously reported results of other authors [42–44], where large perturbation was demonstrated to be necessary for obtaining chaotic motions in the Duffing oscillator with hardening type of nonlinearity.

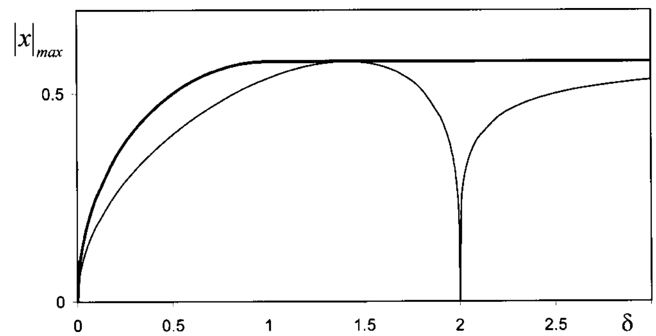

FIG. 4. Size of the asymptotic stability area vs dissipation parameter δ for a soft-mode Duffing oscillator at $\omega_0=1$, $\varepsilon=1$. TLLE analysis and optimized linear coordinate transform (heavy line) compared to the case of the system with diagonalized matrix of the linear part (light line). Note also the difference with hard-mode oscillator (Fig. 2): the size of the stability area is limited from above and independent of dissipation, starting at $\delta=\omega_0=1$.

C. Double-well Duffing oscillator

As the next example, we take a variant of the Duffing oscillator, which possesses a saddle point at the origin in the absence of external forces. We would like now not only to define the maximal stable amplitude of motion, but also compare our results with the predictions for chaos-arising threshold following from the Melnikov theory [45] and conventional stability analysis [30,46]. The equation we analyze has the form

$$\frac{d^2x}{dt^2}+\delta\frac{dx}{dt}-\alpha x+\beta x^3=f(t). \tag{54}$$

In the absence of dissipation and external forcing, i.e., if $\delta=0$ and $f(t)=0$, the system possesses two potential wells, located symmetrically with respect to the origin and centered at $x_0=\pm\sqrt{\alpha/\beta}$. When the parameters controlling damping and external excitation are nonvanishing, the motion is confined within one of the potential wells, until some threshold in amplitude is reached, when the transitions between wells become possible. Chaotic motions exist in this system either as a pair of identical chaotic attractors located symmetrically with respect to the origin when the external excitation is small, or as a single symmetric attractor at larger values of the external force.

Since the method we propose here deals with lower bounds for the amplitude of motion ensuring the asymptotic stability of any trajectory, we do not expect it to be applicable to the "large" attractors embracing both potential wells and consider the dynamics in one of the potential wells only. To make the system (54) consistent with preceding analysis, we apply first the coordinate transform $x\rightarrow x-x_0$, shifting the origin to the center of a potential well. This transforms the Eq. (54) to the form

$$\frac{d^2x}{dt^2}+\delta\frac{dx}{dt}+(2\alpha+3\sqrt{\alpha\beta}x+\beta x^2)x=f(t). \tag{55}$$

After such a transformation, it becomes evident that the only difference of this system with previously considered hard-and soft-mode Duffing oscillators consists in the presence of a quadratic term in the nonlinearity function. This type of nonlinearity is known to be similar in many respects to the case of soft-mode Duffing oscillator [46,47], as it produces the effect of lowering the resonance frequency with the increase in the amplitude of excitation. Another characteristic feature of motion in the potential well described by Eq. (55) is the presence of dynamic asymmetry (or constant shift) of the center of the phase orbit, which manifests itself starting from arbitrarily low values of the force $f(t)$. For example, in the case of harmonic excitation,

$$f(t)=\gamma\cos(\omega t), \tag{56}$$

the approximate solution has to be searched in the form

$$x=A_0+A_1\cos(\omega t+\theta), \tag{57}$$

contrary to the previously considered single-well oscillators, where $A_0=0$ in the first approximation.

The presence of the additional parameter A_0 makes the subsequent analysis a little more complicated, but eventually the resulting equations have a lot in common with those of the soft-mode Duffing oscillator considered above. The functional form of the nonlinear potential is now given by

$$N(x)=3\sqrt{\alpha\beta}x^2+\beta x^3,$$

and the natural frequency of oscillations is defined by the parameter $\omega_0^2=2\alpha$. The stability area can now be introduced in terms of the parameters A_0, A_1 as the range of variation for the variable x: $x\in[A_0-A_1;\,A_0+A_1]$. The maximum and minimum values of the function $V(x)\equiv dN(x)/dx$ can be found from the following equations:

$$\begin{aligned}V_{\min}&=3\beta(A_0-A_1)^2+6\sqrt{\alpha\beta}(A_0-A_1)\\&=[\sqrt{u(\delta-u)}-v]^2-2\alpha,\\V_{\max}&=3\beta(A_0+A_1)^2+6\sqrt{\alpha\beta}(A_0+A_1)\\&=[\sqrt{u(\delta-u)}+v]^2-2\alpha.\end{aligned} \tag{58}$$

Since the inequality $V(x)>-2\alpha$ always holds inside the stability area, we obtain the following limitation on the value of the parameter A_0:

$$A_0\geqslant-\sqrt{(\alpha/3\beta)}(\sqrt{3}-1). \tag{59}$$

The size of the stability area is defined by the parameter A_1. Therefore, the remaining problem consists in maximizing its value by performing an optimal choice of the parameters u, v. The parameter v can be excluded from consideration by combining two of the Eqs. (58) as

$$v=\frac{3A_1(\beta A_0+\sqrt{\alpha\beta})}{\sqrt{u(\delta-u)}}. \tag{60}$$

Substitution of the Eq. (60) in one of the Eqs. (58) leads to the following equation for parameter A_1:

$$A_1^2=\frac{u(\delta-u)}{3\beta}\left(\frac{m(A_0)-\alpha-u(\delta-u)}{m(A_0)-u(\delta-u)}\right),$$

where the notation $m(A_0)\equiv3(\sqrt{\beta}A_0+\sqrt{\alpha})^2$ has been used. The maximization problem for the parameter A_1 with respect to u reduces to the following two cases, depending on the dissipation level and the value of A_0.

(1) If $\delta\geqslant2\sqrt{m(A_0)-[\alpha m(A_0)]^{1/2}}$, then $A_1^{\max}$ does not depend on δ and the maximum is attained at the value of u defined as a root of the equation $\sqrt{u(\delta-u)}=m(A_0)-\sqrt{\alpha m(A_0)}$. Then, we have

$$A_1^{\max}=\frac{1}{\sqrt{3\beta}}\sqrt{\left(\left[\frac{m(A_0)}{\alpha}\right]^{1/2}-1\right)\left(\left[\frac{\alpha}{m(A_0)}\right]^{1/2}+1\right)}. \tag{61}$$

(2) If $\delta\leqslant2\sqrt{m(A_0)-[\alpha m(A_0)]^{1/2}}$, the value of u maximizing the amplitude A_1 is given by $u=\delta/2$, and

$$A_1^{\max}=\frac{1}{\sqrt{3\beta}}\frac{\delta}{2}\sqrt{[m(A_0)-\alpha-\delta^2/4]/[m(A_0)-\delta^2/4]}. \tag{62}$$

For example, in Fig. 5 we plot the dependence of the maximal amplitude $A_1^{\max}$ on the parameters A_0 and δ, at $\alpha=\beta=1/2$. The typical behavior of the curves shown in this figure is similar to that of the soft-mode Duffing oscillator, e.g., the amplitude of stable oscillations is limited from above and does not depend on δ, starting from a certain level of dissipation. Another feature is the increase of the size of asymptotic stability area in the phase space with dissipation in case (2), i.e., when δ is below its critical value.

It is interesting to compare the performance of the proposed method with the results of direct numerical experiments, as well as estimates for the chaotic instability threshold made by means of other techniques, e.g., Melnikov method [31] or the combination of harmonic balance method with Floquet-type analysis [46,48]. Usually, in the framework of these methods, the analysis is conducted in terms of control parameters describing the external force, such as, the amplitude and frequency of a harmonic excitation or similar characteristics of a quasiperiodic forcing. Following [45] and [46] we take the external force in the form of harmonic excitation given by Eq. (56) and search the approximate solution of Eq. (55) in the form (57). The application of the harmonic balance method [30] gives the solution in the following form [46]:

$$A_0=\left(\frac{\alpha}{\beta}-\frac{3}{2}A_1^2\right)^{-1/2}-\left(\frac{\alpha}{\beta}\right)^{1/2}, \tag{63}$$

$$\left[\left(2\alpha-\omega^2+\frac{15}{4}\beta A_1^2\right)^2+\delta^2\omega^2\right]A_1^2=\gamma^2. \tag{64}$$

Equation (63) allows us to exclude the constant bias A_0 from the asymptotic stability analysis by substituting it in Eq. (58) and obtain the equation for the border of stability area in terms of A_1 only. Some algebraic transformation of Eqs. (59)–(62), taking into account the constraint (63), results in the conclusion that the maximal stable amplitude A_1 is limited from above by the value A^*, where A^* is found as the minimal positive root of the equation

$$2\alpha-\frac{3\beta}{2}A^{*2}-6\beta A^*\left(\frac{\alpha}{\beta}-\frac{3}{2}A^{*2}\right)^{1/2}=0. \tag{65}$$

Then, depending on the value of the dissipation parameter, we have two cases to consider:

(1) If $\delta>\delta_{\rm cr}$, then the maximal stable amplitude does not depend on the dissipation level, and $A_1^{\max}=A^*$.

(2) If $\delta<\delta_{\rm cr}$, then the maximal size $A_1^{\max}$ of the area of asymptotic stability can be derived from the equation

$$\sqrt{2\alpha-(3\beta/2)(A_1^{\max})^2+6\beta A_1^{\max}\left(\frac{\alpha}{\beta}-\frac{3}{2}(A_1^{\max})^2\right)^{1/2}}-\sqrt{2\alpha-(3\beta/2)(A_1^{\max})^2-6\beta A_1^{\max}\left(\frac{\alpha}{\beta}-\frac{3}{2}(A_1^{\max})^2\right)^{1/2}}=\delta, \tag{66}$$

where

$$\delta_{cr}=\sqrt{2\alpha-(3\beta/2)A^{*2}+6\beta A^*\left(\frac{\alpha}{\beta}-\frac{3}{2}A^{*2}\right)^{1/2}}.$$

Figure 6 shows the result of calculations for the border of stability area by means of our method. For the chosen values of the parameters $\alpha=\beta=\frac{1}{2}$, the magnitude of A^* can be found from Eq. (65) as $A^*\cong 0.335$, which establishes the critical value of dissipation at the level $\delta_{\rm cr}\cong 1.354$. Choosing δ as, e.g., $\delta=1$, one can obtain the limiting value for the maximal stable amplitude as $A_1^{\max}\cong 0.296$ at $A_0\cong -0.068$. The solid line in Fig. 6 corresponds to the result of numerical integration of the Eq. (54), and indicates the locus of points on the ω-γ plane where the amplitude of oscillations A defined as $A=(x_{\max}-x_{\min})/2$ reaches the value of $A_1^{\max}\cong 0.296$. The area of chaos is located well above this line, as could be expected from the theoretical analysis given above. It is also evident that the TLLE method strongly underestimates the position of chaotic area, since it predicts, in fact, the onset of any type of instabilities (not just chaotic motions), like period-doubling or saddle-node bifurcations, some of which are known to appear at much lower levels of external excitation than chaotic attractors. The borderline of instability can be also obtained analytically by utilizing the harmonic balance method that links the amplitude of stable oscillations to the parameters γ and ω. The line calculated by substituting the value $A_1^{\max}\cong 0.296$ to the Eq. (64) is also plotted in Fig. 6 for the sake of comparison. One can notice good agreement of this prediction with the results of the direct numerical integration, although there is a certain discrepancy at small values of the excitation frequency. It should be, however, noted that this deviation between the curves comes solely from inefficiency of the harmonic balance method at low frequencies and high values of dissipation, resulting from inaccuracy of the approximation (57) in this part of the control parameters space.

In order to compare the prediction of the chaos threshold given by Eqs. (61)–(64) with Melnikov theory, we also plot in Fig. 6 the critical line where the homoclinic structures associated with the saddle point at the origin of the unperturbed system are expected to appear. We use for this purpose the explicit formula obtained in Refs. [32], [45],

$$\gamma_c=\frac{2\delta\alpha\sqrt{\alpha}}{3\omega\sqrt{2\beta}}\cosh\left(\frac{\pi\omega}{2\sqrt{\alpha}}\right). \tag{67}$$

Note that at the chosen (rather high) value of dissipation parameter the Melnikov method fails to provide the correct location of the chaos area and gives an absolutely misleading prediction indicating the threshold for chaotic motions at much higher levels of the external force than those where strange attractors actually appear (see also Ref. [46]).

Another method of predicting the onset of chaos in this system has been reported in Ref. [46]. It places the chaotic area between two critical values of the frequency on the ω-γ plane: the vertical tangent to the amplitude response curve defined by the Eq. (64) and the first period doubling bifurcation. Although this criterion gives quite accurate prediction for the border of chaotic zones at small values of dissipation parameter, it loses the accuracy at higher levels of dissipation especially in the low-frequency part of the ω-γ plane. Here we use an earlier version of the same method [49] that demonstrates better performance at high δ values. The empirical criterion for chaos has been formulated for $\alpha=\beta=\frac{1}{2}$ as

$$1-\tfrac{3}{2}\sqrt[3]{(15/4)\gamma^2}\leqslant\omega^2\leqslant\tfrac{1}{2}\left(1-\delta^2+\sqrt{\delta^4-2\delta^2+15\gamma^2}\right), \tag{68}$$

where the chaotic area is located between the frequencies of the vertical tangent to the response curve and maximal amplitude of the response calculated in accordance with Eq. (64). The results of calculation using the inequalities (68) are also plotted in Fig. 6. Apparently, this approach gives good prediction for the location of chaotic areas on the ω-γ plane, even in its low-frequency part and at quite high value of the dissipation parameter ($\delta=1$). It is interesting that, although this method is, in fact, an empirical technique based on certain subjective assumptions on the possible location of chaos area in the control parameters space, it implicitly uses certain information on the asymptotic stability of motion. Indeed, the criterion (68) considers the *amplitude* of motion as the most important indicator of incipient instability, and predicts

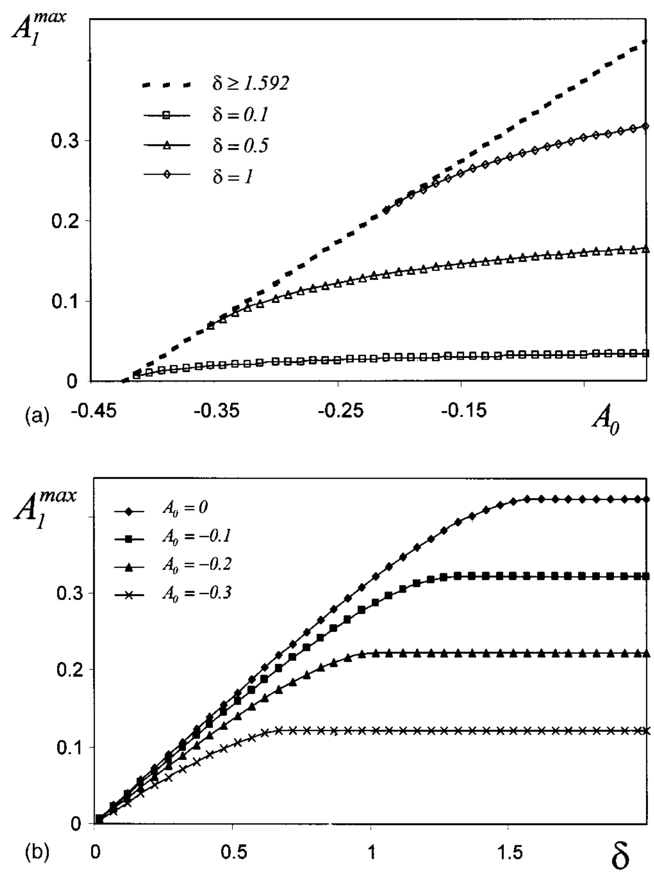

FIG. 5. The size of asymptotic stability area in the double-well Duffing oscillator (54) vs constant shift of solution due to asymmetry of periodic orbits (a) and dissipation level (b) at $\alpha=\beta=\frac{1}{2}$.

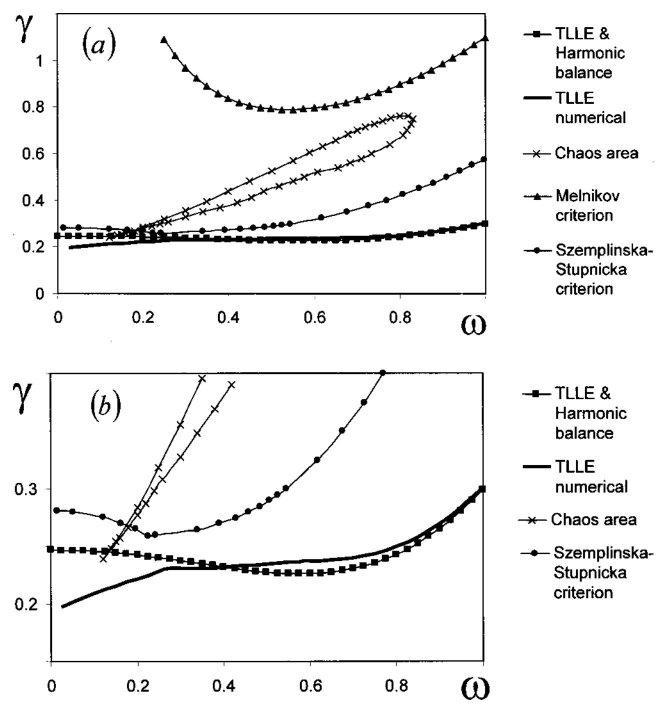

FIG. 6. State diagram of the double-well Duffing oscillator (54) at $\alpha=\beta=\frac{1}{2}$; $\delta=1$ (a) and its blow-up (b).

the appearance of chaotic attractors within certain range of the x coordinate, where expanding directions exist in the vicinity of saddle-type orbits created by the external force. Note, however, that unlike the approach proposed in the present paper, this method has certain limitations restricting its applicability in many situations of practical interest. For example, it cannot be used when it is necessary to predict the onset of chaos in a nonlinear oscillator being excited by the external force containing more than one harmonic component or when a reliability of the prediction is of crucial importance.

D. Pendulum oscillator

As the last example we take a classic nonlinear dynamical system, the externally driven damped pendulum. Starting from the works of Huygens more than 300 years ago, this oscillator has been at the focus of enormous interest due to its apparent simplicity, richness of dynamical behavior, and importance of applications, such as, e.g., resistively shunted Josephson junction [50], where it has been used as an adequate mathematical model. In dimensionless form, the equation of motion for this system reads

$$\frac{d^2x}{dt^2}+\delta\frac{dx}{dt}+\sin(x)=f(t), \tag{69}$$

where x is the angle of elevation of the pendulum, dx/dt is its angular velocity, δ is the damping term responsible for the decay of oscillations in the absence of external force, and $f(t)$ is the driving torque. Contrary to the previously considered Duffing oscillator, the potential function for this oscillator has a periodic character, i.e., consists of infinite number of potential wells separated by the distance of 2π in the angular coordinate x. Near the bottom of each of the wells, the behavior of this system is similar to that of the soft-mode Duffing oscillator, with the only exception that both the degree of nonlinearity and the parameter of natural frequency of oscillations are now controlled by the sin(x) function, and, therefore, cannot be tuned independently. In order to use the results obtained in the previous section for the oscillator of type (33), we add and subtract the term $\omega_0^2 x$ from the left-hand side of the Eq. (69) and have for the case of $\omega_0^2=1$; $\varepsilon=1$:

$$N(x)=\sin(x)-x;\quad V(x)=\cos(x)-1.$$

Maximal and minimal values of the coordinate x are now defined by the equation

$$V_{1,2}=[\sqrt{u(\delta-u)}\mp v]^2-1,$$

where, again, u and v have to be chosen as maximizing the amplitude of stable motions around the origin. Following the same reasoning as the one we used in the case of Duffing oscillator, the critical value of dissipation parameter $\delta_{\rm cr}=1$, which separates two qualitatively different types of behavior for oscillations located in a symmetric area around the origin, can be obtained.

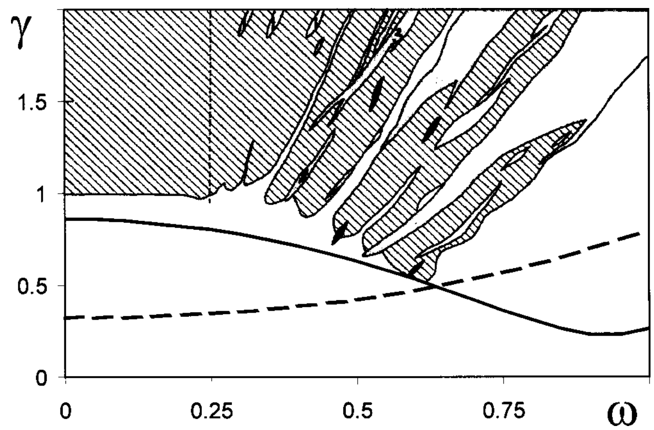

FIG. 7. Comparative performance of the proposed method, i.e., the combination of harmonic balance and TLLE analysis (solid line), and Melnikov criterion (dashed line) for the pendulum oscillator (71). The areas of chaotic behavior are hatched (from Ref. [51]).

(1) If $\delta<\delta_{cr}$, then the maximal displacement is defined by

$$|x|_{max}=\cos^{-1}[(1-\delta)^2]. \tag{70a}$$

(2) $\delta>\delta_{cr}$, then the size of stability area does not depend on dissipation and is limited by the value of

$$|x|_{max}=\frac{\pi}{2}. \tag{70b}$$

In order to check the efficiency of the proposed criterion and compare it to the results of previous works [51–53], we consider the case of harmonic excitation of type (56) and use the following formula for an approximate solution of Eq. (69):

$$x=A\cos(\omega t+\varphi), \tag{71}$$

where the amplitude A can be found as a root of the equation [51]

$$[2J_1(t)-\omega^2 A]^2+(A\omega\delta)^2=\gamma^2. \tag{72}$$

For any given value of the dissipation level, Eqs. (70) specify the upper bound of the amplitude of oscillations with the guaranteed stability. By substituting the value of maximal stable amplitude defined by Eq. (70) to the Eq. (72), one can obtain the borderline of the asymptotic stability area expressed in the explicit form $\gamma(\omega)$. Extensive numerical analysis of the oscillator (69) has been performed in Ref. [51] for the value of $\delta=0.25$, and the locus of chaotic areas on the plane of control parameters ω-γ has been established. In Fig. 7 we show the results of [51] together with the curve delimiting the area of asymptotic stability found from Eqs. (70a) and (72) at $\delta=0.25$. Apparently, the stability analysis provides rather good prediction as for the onset of chaotic oscillations, especially in the region of small frequencies of external excitation. For the sake of comparison, we also show on the same plot the curve corresponding to Melnikov criterion [41],

$$\gamma_{cr}=\frac{4\delta}{\pi}\cosh\left(\frac{\omega\pi}{2}\right),$$

which also gives a good estimate for the threshold of chaos in pendulum oscillator at the chosen value of dissipation parameter. Therefore, we arrive at a conclusion that for this system the two criteria work in a complementary manner, predicting the onset of chaotic motions in different frequency bands.

VI. DISCUSSION

We have proposed an analytic criterion for predicting the onset of chaotic motion in a broad class of nonautonomous damped nonlinear oscillators. We suppose it can be used as a first step in investigating complicated nonlinear regimes that can arise in oscillators subject to the external perturbation. In fact, the method allows obtaining the border of stability area in the phase space or, being combined with another technique, e.g., the harmonic balance procedure, in the space of control parameters. Although it, as a rule, underestimates the actual position of the threshold of chaos, in some situations it performs better than other existing techniques such as Melnikov method or Floquet-type stability analysis of periodic attractors. Our approach is expected to be especially useful in situations when chaotic behavior is an undesired effect, and the problem consists in finding the area in the control parameter space where the motion is stable and by no means chaotic. From this viewpoint, we provide the stability criterion for nonlinear dynamical systems which guarantees the absence of an additional noise source coming from the chaotic dynamics.

It has been recently recognized that in many oscillatory systems the threshold of chaos may be strongly dependent on the frequency content of the external excitation. As it was shown, e.g., in Refs. [29], [54], the change of harmonic to bifrequency excitation in an equation of class (33) results in considerable lowering of the chaos onset in the intensity of the external force. A natural question stems from these findings: what is the lowest possible level of excitation that can result in chaotization of motion? As we have demonstrated with several examples of nonlinear oscillators, the analysis of asymptotic stability in terms of TLLE allows us to answer this question and to estimate the maximal stable amplitude of motion, and thus provides a necessary condition for chaotic motion and any other bifurcation as well. We would like to stress that the method we propose is independent of the type of external force and dimensionality of the dynamical system, therefore, it yields a fundamental limit for chaotic instability to appear in a broad class of nonlinear dynamical systems.

Another important motivation for applying this particular method to the analysis of dynamical systems is that it provides a rigorous necessary condition for chaotic (or Lyapunov-type) instability to appear. We would like to note that despite considerable efforts undertaken to formulate (in terms of control parameters) necessary and/or sufficient conditions for the emergence of chaotic attractors in nonlinear systems, there seems to be no universal criterion existing at the moment. In the situation when the prediction of chaotic motion is necessary, the Melnikov method is commonly used. This approach provides an estimate of the distance between perturbed stable and unstable manifolds for a particular saddle-type orbit existing in the unperturbed system. If, in the presence of perturbation, this distance can vanish at some value of controls, this means an intersection of stable and unstable manifolds and presence of geometrically complex structures in the phase space. This analysis allows us to calculate the parameter values where the homoclinic structures appear indicating the possibility for chaotic motions to be formed. Apparently, this method gives neither sufficient nor necessary condition for chaos. Indeed, it does not provide a sufficient condition, because the method cannot guarantee that the homoclinic structure, once emerged, becomes attractive and forms a strange attractor. It does not constitute a necessary condition for chaos either, because of the presence of multistability in any nonlinear oscillatory system. By the term multistability we mean coexistence of several attractors in the phase space at fixed values of all the controls. Typically, each of the attractors occupies a well-separated area in the phase space and, as the control parameters change, every attractor may undergo various bifurcation sequences independently of the others. Some of them may become chaotic at much lower levels of perturbation compared to those predicted by Melnikov's criterion.

Of course, our understanding of the necessity depends on definition. We suppose that the following statement should constitute a basis for a definition of the necessary condition for chaos, especially in view of its importance for engineering and other applications where chaos is considered as an undesired effect like, e.g., a source of additional noise, etc. *A necessary condition of chaos guarantees the absence of chaotic regimes in the system in the case it is not satisfied*. It is evident that from this viewpoint the Melnikov method cannot be considered as a proper one, since it can only guarantee the absence of chaotic motions associated with a *given* saddle state, and not *any* chaotic attractor [32,55].

The method proposed in the present paper estimates the maximal value of oscillation amplitude below which chaos cannot occur at all. Accordingly, the larger amplitude means the possibility for chaos to appear. We assert that this criterion can be used as a necessary condition for chaos in accordance with the definition given above. As far as amplitude of oscillations is concerned, it is a rigorous analytic criterion, without any approximation used at any stage of estimating the size of the stability area in the phase space. In order to obtain the border of instability in terms of control parameters, e.g., amplitude and frequency of external force, some approximation is necessary for establishing the relation between those parameters and size of the attractor.

Although in this paper we restricted our consideration by the systems with three-dimensional phase space, the method can be also utilized for specifying stability threshold in high-dimensional dynamical systems as well. The straightforward way of performing such an analysis consists in applying the linear coordinate transform that makes the matrix of the linear part of the problem diagonal or block diagonal. Then, the equation for the border of the stability area can be obtained from the explicit equation of type (9) for the largest TLLE.

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